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Abstract

This paper is concerned with the problem of deciding how much scholarship aid to offer to each financially needy applicant who has been admitted to the freshman class of an institution of higher learning. Two quadratic programming models are developed under different assumptions with regard to non-scholarship aid. Both models relate the total expected cost of freshman scholarship support to the expected number of scholarship students and a measure of their expected quality. The models are intended to assist student aid administrators in determining reasonably successful freshman scholarship offer policies.

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Mathematical Programming Models for Determining Freshman Scholarship Offers*

by Leon S. White

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1. Introduction

This paper is concerned with the problem of deciding how much scholarship aid to offer to each financially needy applicant who has been admitted to the freshman class of an institution of higher learning. It is assumed that the admission decisions that determine the members of the admitted freshman class are made prior to all decisions regarding scholarship awards and are based solely on evidence of academic fitness and professional promise. The amount of financial resources available to any applicant is not considered in the admission decision. It is also assumed that the preferred solution to the problem, that of offering each needy admitted applicant as much scholarship aid as he requires, is infeasible.

The problem of determining scholarship award offers to admitted freshman applicants with financial need (hereafter usually referred to as "needy admits") is faced annually by student aid administrators at most colleges and universities throughout the country. The particular models described in this paper were developed to assist the administrators of the Student Aid Center at the Massachusetts Institute of Technology; however, the results presented should be of more general interest and use.

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The group of freshman admits who have been classified as financially needy constitute the total population of admitted applicants to which our models pertain. Admits who are considered for scholarship awards on grounds other than financial need are not included. We shall assume that an admitted applicant's financial need is determined by two estimates: (i) an estimate of a reasonable school and personal expense budget for the normal academic year and (ii) an estimate of the money the admit and his parents can be expected to contribute toward meeting his expenses. The school and personal expense budget, hereafter called the standard budget, is determined by adding up the costs of tuition, medical insurance, room and board, an allowance for books and materials, and a second allowance for miscellaneous expenses. The estimates of the contribution expected from the admit and his parents is based primarily on an assessment of the "Parents Confidential Statement" of personal finances (submitted through the College Scholarship Service of the College Entrance Examination Board) and on an estimate of the summer earnings capability of the admit. An admit's need, if any, is defined as the difference between the standard budget and the contribution expected from him and his parents.

After the financial condition of each member of the admitted freshman class has been determined, the class of all admits can be divided into two groups: the need group and the no-need group. Members of the need group may receive financial aid from non-university and/or university sources. It is assumed that if a needy admit is offered useable scholarship aid from one or more non-university sources, e.g. from the National

Merit Scholarship Program, and if the admit decides to matriculate, he will accept all outside awards offered to him. Thus, his need in so far as the university is concerned is reduced by the amount of outside aid.

At the point in time when a scholarship offer policy must be decided, the decisionmakers who will determine this policy should have some information about the amounts of aid that have been offered to members of the need group from non-university sources. In addition, these administrators should have a backlog of information on non-university scholarships offered to needy admits in past years. We shall assume that on the basis of both current and historical information on outside support, the policymakers can accurately estimate the amount of non-university financial aid that will be offered to each needy admit, prior to their determination of a scholarship offer policy. Thus, the scholarship award offered by the university to any member of the need group will provide an amount of money equal to some fraction of his residual need, i.e. the difference between his total need and the amount of scholarship aid he is expected to receive from non-university sources. In cases where the residual need is not met in full by a scholarship, we shall assume that sufficient non-scholarship aid, i.e. a loan and/or term-time work, will be offered to make up the difference.

The success of any scholarship offer policy may be measured, a posteriori, in terms of the number of needy admits that enroll, their quality, and the amount of scholarship aid that they accepted. The number

that enroll will usually be compared with a target figure and if the two numbers are close to each other, the policy can be considered a success with regard to yield.

Success with regard to quality may be more difficult to measure. However, since we have assumed that all admissions decisions precede financial aid considerations it follows that at a minimum the quality (i.e. academic fitness and professional promise) of any group of needy admits that enroll has already been judged admissible. Nevertheless, the decisionmakers may still want to take into account differences in quality within the total need group in developing a scholarship offer policy. One way to do this would be to develop criteria by which to rank all the members of the need group from first to last with regard to quality. Then a policy could be evaluated as to its quality yield in terms of the number of enrollees from among the higher ranked members of the need group.

The monetary success of a scholarship offer policy may be measured by comparing the number of dollars actually spent on aid with the number of dollars budgeted. If the budgeted figure is exceeded by a non-trivial amount, the implemented policy must be judged a financial failure--even though it may have been an obvious success when judged by the number of needy enrollees and their quality.

In developing mathematical models to assist scholarship aid administrators in choosing an offer policy, we shall be guided by the a posteriori measures of numerical yield, quality, and cost described above. Our models, however, must take into account the fact that there is no way

to predict with certainty how any needy admit will respond to a particular offer of financial aid, and, hence, there is no way, a priori, to precisely forecast the results of any proposed policy. Therefore, we must begin by developing a careful description of the scholarship offer and acceptance process that includes the probabalistic aspects of the situation. This description, or descriptive model, can then be used as a basis for developing optimization models by which to evaluate alternative policies.

The next section of this paper is devoted to the development of a descriptive quantitative model. A general optimization model is presented in section 3. A special case of this general model is formulated in section 4. This special case has the form of a quadratic program, and, consequently, it is amenable to a computer solution. In the fifth section a particular version of the quadratic program is formulated in which an upper bound is placed on the non-scholarship component included in any university financial aid package offered to a needy admit. The last section is devoted to a brief discussion of possibilities for future research.

2. A Descriptive Model

Consider, now, a need group consisting of m admits. Suppose that the admits are identified and ordered as to their quality by the index i, (i=1,2,...,m), such that the quality of admit 1, (i=1), is superior or equal to the quality of admit 2, (i=2), the quality of admit 2 is superior or equal to the quality of admit 3, (i=3), and so on.

Let the demonstrated need of admit i be given by the product $n_i B$, $(0 < n_i \leq 1)$, where B is the amount of the standard budget. And let $e_i B$, $(e_i \geq 0)$, denote the total amount of scholarship aid offered to admit i by external (non-university) sources. Furthermore, let $u_i B$ represent the amount of university scholarship aid from university sources that will be awarded to admit i if he enrolls, where for i=1,2,...,m,

$$u_i = 0, \text{ if } e_i \geq n_i$$
$$0 \leq u_i \leq n_i - e_i, \text{ if } e_i < n_i . \tag{2.1}$$

Any vector $\underline{u} = (u_1, u_2, \ldots, u_m)$ that satisfies (2.1) describes a possible scholarship offer policy.

Finally, let $\{\tilde{z}_i : i=1,2,\ldots,m\}$ denote a set of indicator random variables, where $(\tilde{z}_i=1)$ represents the event that admit i enrolls and $(\tilde{z}_i=0)$ the event that he does not; and define the conditional probability function $f_i(z_i)$ as,

$$f_i(\tilde{z}_i) = P(\tilde{z}_i = z_i \mid B, n_i, e_i, u_i), \qquad (i=1,2,\ldots,m). \tag{2.2}$$

Thus, $f_i(1)$ is the probability that admit i enrolls given that he is offered $e_i B$ dollars in scholarship aid from non-university sources, $u_i B$ dollars

in additional scholarship aid from the university, he is i^{th} ranked with regard to quality, and his total demonstrated need is $n_i B$.

Using the probability function (2.2) we can characterize the expected number of enrolling needy admits for any policy $\underline{u}$ by the equation,

$$E\left(\sum_{i=1}^{m} \tilde{z}_i\right) = \sum_{i=1}^{m} f_i(1). \tag{2.3}$$

Moreover, letting α be any number between 0 and 1, and defining $[\alpha m]$ as the largest integer less than or equal to αm, we can characterize the expected number of enrolling needy admits from the top 100α % of the need group, i.e., from among the admits with quality indices $1,2,\ldots,[\alpha m]$ by the equation,

$$E\left(\sum_{i=1}^{[\alpha m]} \tilde{z}_i\right) = \sum_{i=1}^{[\alpha m]} f_i(1). \tag{2.4}$$

And, finally, we can describe the expected cost of any policy $\underline{u}$ by the equation,

$$E\left(\sum_{i=1}^{m} u_i B\tilde{z}_i\right) = B \sum_{i=1}^{m} u_i f_i(1). \tag{2.5}$$

The equations (2.3), (2.4), and (2.5) are viewed collectively as a descriptive model of the scholarship offer and acceptance process that relates the inputs,

(i) the quality ranking index i

(ii) the demonstrated need vector $\underline{n} = (n_1, n_2, \ldots, n_m)$

(iii) the non-university scholarship aid vector

$$\underline{e} = (e_1, e_2, \ldots, e_m)$$

(iv) the conditional probability function vector

$$\underline{f} = (f_1(1), f_2(1), \ldots, f_m(1))$$

(v) the standard budget B

and (vi) the policy parameter α ,

to expected yield, quality, and cost related outputs, for a given vector of policy variables $\underline{u}$.

3. General Optimization Model

The descriptive model, (2.3), (2.4), and (2.5), provides three a priori measures of performance for any proposed scholarship offer policy:

1. The expected number of enrolling needy admits, $T(\underline{u})$,
2. The expected number of enrolling need admits from among the top 100α % of the need group, $Q(\underline{u},\alpha)$
3. The expected cost, $C(\underline{u})$.

The values of $T(\underline{u})$ and $Q(\underline{u},\alpha)$ may be viewed as measures of the benefits of any policy $\underline{u}$, while $C(\underline{u})$ is reflective of its cost. The general problem of finding an optimal scholarship offer policy may then be formulated either in terms of a criterion which compares the benefits of alternative policies all of which are subject to the same constraint on cost, or in terms of a criterion which compares the cost of alternative policies all of which are subject to the same minimum standards with regard to benefits.

The criterion to be used in formulating the general optimization model is of the second type. It is specified as follows: An optimal policy $\underline{u}^*$ must satisfy the following conditions,

1. $T(\underline{u}^*) = T$, a target value.
2. $Q(\underline{u}^*,\alpha) \geq \beta T$, $0 \leq \beta \leq [\alpha m]/T$
3. $C(\underline{u}^*) = \min C(\underline{u})$ for all policies that satisfy conditions 1 and 2.

The general optimization model (and problem) corresponding to these conditions

can be written as follows: Given the quality ranking index i; the input parameters $\underline{n}$, $\underline{e}$, $\underline{f}$, and B; and the policy parameters α, β, and T, find a scholarship policy $\underline{u}$ to

$$\text{Minimize:} \quad C(\underline{u}) = B \sum_{i=1}^{m} u_i f_i(1)$$

$$\text{Subject to:} \quad \sum_{i=1}^{m} f_i(1) = T$$

$$\sum_{i=1}^{[\alpha m]} f_i(1) \geq \beta T \tag{3.1}$$

$$u_i \leq \max (n_i - e_i, 0), \qquad (i=1,2,\ldots,m)$$

$$u_i \geq 0, \qquad (i=1,2,\ldots,m).$$

The general model (3.1) is conceptually useful, however, it cannot be manipulated numerically until a functional form is postulated for the elements of the conditional probability vector $\underline{f}$. In the next section we develop a linear form which allows (3.1) to be reduced to a quadratic program.

4. A Quadratic Programming Formulation

Recall that $f_i(1)$ was defined as the probability that needy admit i enrolls given that he is offered e_iB dollars in scholarship aid from non-university sources, u_iB dollars in additional scholarship aid from university sources, he is i^{th} ranked with regard to quality and his total demonstrated need is n_iB. Let $f_i(1)$ be written as $f_i(1 \mid u_i)$ to emphasize its dependence on u_i. Then we postulate the following behavior for $f_i(1 \mid u_i)$ for fixed values of i, n_i, and e_i:

1. For u_i close to zero, $f_i(1 \mid u_i)$is relatively constant.
2. For u_i close to $n_i - e_i$, $f_i(1 \mid u_i)$ is relatively constant.
3. $f_i(1 \mid n_i - e_i) - f_i(1 \mid 0) > 0$.
4. For u_i sufficiently greater than 0 and less than $n_i - e_i$, $df_i(1 \mid u_i) / du_i$ is relatively constant.

These four conditions imply a functional form for $f_i(1 \mid u_i)$ such as is pictured in Figure 4.1.

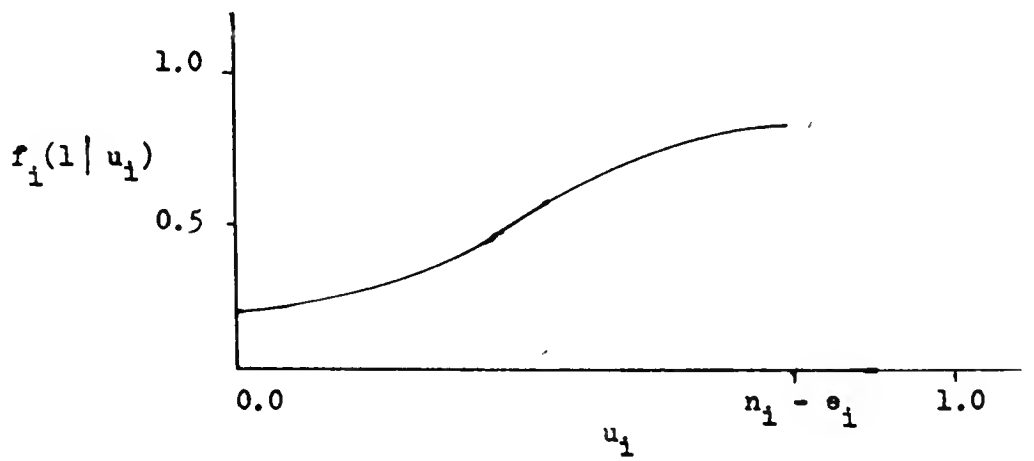

Figure 4.1

Assuming the validity of conditions 1 through 4 above, it does not seem unreasonable to postulate a linear form for $f_i(1 \mid u_i)$ as a satisfactory approximation to its actual shape. Thus, we shall let

$$f_i(1 \mid u_i) = a_i + b_i u_i \qquad 0 \le u_i \le n_i - e_i. \tag{4.1}$$

We further assume that the parameters a_i and b_i can be estimated for all members of the need group. The parameter a_i represents the probability that needy admit i will enroll given that he is offered max $(n_i - e_i, 0)$ dollars in non-scholarship aid over and above the scholarship aid offered from outside sources. Letting

$$d_i = \max (n_i - e_i, 0)$$

and considering a_i as a function of d_i and i it would seem possible that a set of equi-probability curves such as illustrated in Figure 4.2 could be developed from which to estimate values for any set of a_i's.

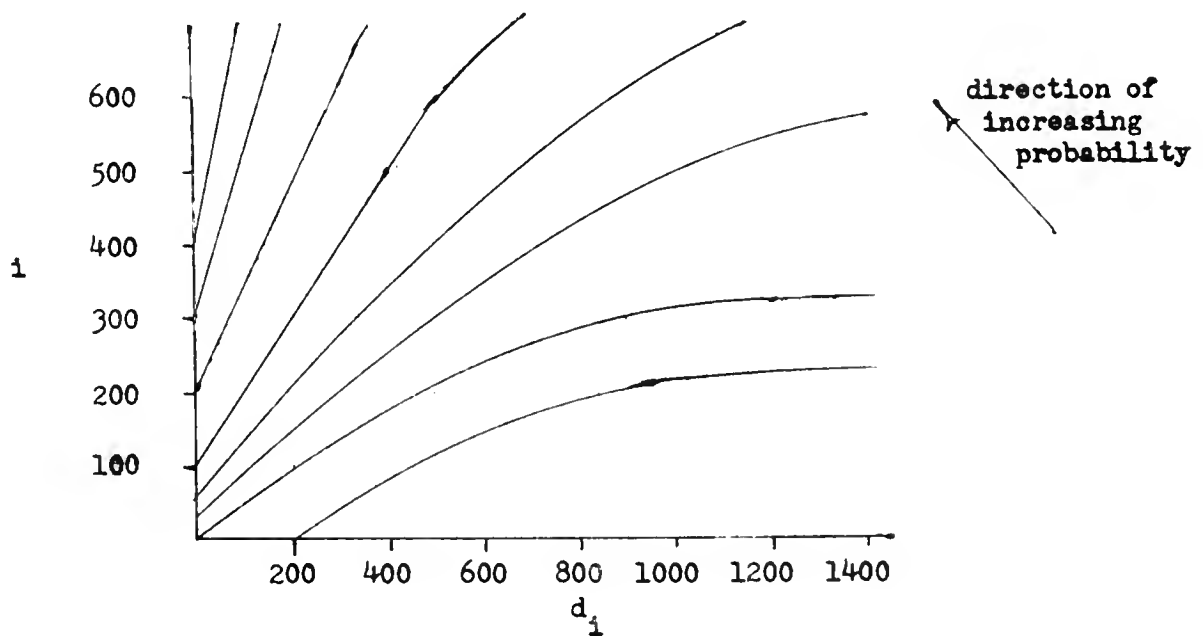

Figure 4.2

If a_i were considered to be a function of d_i only, then it might be reasonable to assume that a_i and d_i have the functional relationship

$$a_i = v - wd_i,$$

in which case the estimation problem is reduced to finding values of v and w.

The parameter b_i estimates the increase in $f_i(1)$ for a unit increase in the policy variable u_i. In general, b_i will also be a function of d_i and i.

When the linear form (4.1) is substituted into the general model (3.1) the following quadratic program results: Given the quality ranking index i; the input parameters $\underline{n}$, $\underline{e}$, $\underline{a}$, $\underline{b}$, and B; and the policy parameters α, β, and T, find values for the policy variables $\underline{u}$ to,

$$\text{Minimize: } C(\underline{u}) = B\left[\sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{m} b_i u_i^2\right]$$

$$\text{Subject to: } \sum_{i=1}^{m} b_i u_i = T - \sum_{i=1}^{m} a_i$$

$$\sum_{i=1}^{[\alpha m]} b_i u_i \geq \beta T - \sum_{i=1}^{[\alpha m]} a_i \qquad (4.2)$$

$$u_i \leq \max\,(n_i - e_i, 0), \qquad (i=1,2,\ldots,m)$$

$$u_i \geq 0, \qquad (i=1,2,\ldots,m).$$

An efficient computer program to solve problems of the form (4.2) can be developed readily from a standard quadratic programming code. Hence,

if the assumptions and criteria underlying the formulation of (4.2) are accepted, the program can be used to explicitly determine a mathematically optimal solution to the question of how much scholarship aid to offer to each needy freshman admit.

The quadratic program (4.2) can also be used parametrically as an experimental tool to determine relationships between the cost of an optimal policy C^* and values of the policy parameters α, β, and T. For example, suppose that α and β are fixed, and (4.2) is solved repeatedly for a range of potential target values T. As a result of these computations a graph of $C^*(T)$ could be plotted and used to determine a reasonable (C^*,T) combination in cases where a value of T was not fixed in advance. Similar marginal analyses could be carried out for the functions $C^*(\alpha)$ and $C^*(\beta)$.

5. A Quadratic Program for the Case Where an Upper Limit Is Placed On Non-scholarship Aid.

In this section we consider the situation in which an upper limit is placed on the amount of non-scholarship aid which may be offered to any needy admit. Let r_i denote the amount of non-scholarship aid offered to admit i by the university. Then, for all i=1,2,...,m,

$$\begin{aligned} r_i &= \max(n_i - e_i, 0) - u_i \geq 0 \\ &= d_i - u_i \geq 0. \end{aligned} \tag{5.1}$$

Now suppose that r denotes the maximum fraction of the standard budget that can be offered to a needy admit in the form of non-scholarship aid. Then

$$r_i \leq \min\,(r, d_i), \qquad (i=1,2,\ldots,m). \tag{5.2}$$

When the equations (5.1) and the inequality (5.2) are introduced into the quadratic program (4.2), the following program results: Given the quality index ranking i; the input parameters $\underline{n}$, $\underline{a}$, $\underline{b}$, and B; and the policy parameters α,β, T, and r, find values for the policy variables $\underline{r} = (r_1, r_2,\ldots,r_m)$ to:

$$\text{Minimize:} \quad C(\underline{r}) = B\Big[\sum_{i=1}^{m} a_i(d_i - r_i) + \sum_{i=1}^{m} b_i(d_i - r_i)^2\Big]$$

$$\text{Subject to:} \quad \sum_{i=1}^{m} b_i(d_i - r_i) = T - \sum_{i=1}^{m} a_i$$

$$\sum_{i=1}^{[\alpha m]} b_i\,(d_i - r_i) \geq \beta T - \sum_{i=1}^{[\alpha m]} a_i \tag{5.3}$$

$$r_i \leq \min\,(r, d_i), \qquad (i=1,2,\ldots,m)$$

$$r_i \geq 0, \qquad (i=1,2,\ldots,m).$$

(Note: In order to help insure feasibility it might be advisable to change the yield equation to a greater than or equal to inequality.)

The quadratic program (5.3) presents experimental possibilities for parametric analysis similar to those discussed in connection with the program (4.2). For example, for fixed values of α, β, and T, (5.3) could be solved for different values of r, thus supplying data for a graph to relate r to the cost of an optimal policy.

6. Discussion

The models presented in this paper are intended to assist student aid administrators in determining reasonably successful freshman scholarship offer policies. It is expected that they will serve best as experimental tools to allow more thorough investigations than are now possible of relationships between input parameters, policy parameters, and output measures. Furthermore, they will focus particular attention on the marginal trade-offs between cost, numerical yield, and quality.

It should be noted that the quality constraint in models (4.2) and (5.3) is in fact a special case of a general constraint of the form:

$$\sum_{i \in S} f_i(1) \geq \beta(S)T, \tag{6.1}$$

where S is some well defined subset of the need group and $\beta(S)T$ is a lower bound on the expected number of enrolling needy admits from S. Thus, additional constraints of the form (6.1) could be included to take account of characteristics such as parent's home state, parent's income level, the admit's sex, etc. For example, the subset S might include all needy admits whose parents live in the state where the university is located.

It should also be noted that the models do not take into account the likelihood that some scholarship funds will only be available to needy admits with certain special qualifications. This complicating factor, though important, would not be expected to seriously affect the usefulness of our models.

In the next stage of this research effort we plan to undertake empirical studies to evaluate the practicability of implementing models such as (4.2) and (5.3). We shall also carry the analysis in another direction towards a study of the total undergraduate student aid program.

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